Criticality of natural absorbing states

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We study a recently introduced ladder model that undergoes a transition between an active and an infinitely degenerate absorbing phase. In some cases the critical behavior of the model is the same as that of the branching-annihilating random walk with $N \geqslant 2$ species both with and without hard-core interaction. We show that certain static characteristics of the so-called natural absorbing states develop power-law singularities that signal the approach of the critical point. These results are also explained using random-walk arguments. In addition to that we show that when dynamics of our model is considered as a minimum-finding procedure, it has the best efficiency very close to the critical point.

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I. INTRODUCTION

In recent years, a large amount of theoretical works [1] has been devoted to the study of phase transitions in model with absorbing states. One aim of these studies is to characterize the possible universality classes. Only few exact results can be obtained. Accordingly, numerical simulations, particularly Monte Carlo simulations and finite size scaling methods, have been widely used [2].

Usually, Monte Carlo method is used to simulate a model in its active phase, where various steady-state characteristics can be measured. When the system approaches a critical point these characteristics develop some critical singularities. But the critical behavior can be also studied from an absorbing phase. In this case, however, one usually studies certain dynamical properites of the model that exhibit critical singularities. For example, the average time needed to reach an absorbing state diverges when approaching the critical point. When appropriately defined, all critical exponents can be defined through some dynamical singularities in an absorbing phase.

Is it possible to infer some information about criticality of the model from the structure of absorbing state(s)? Very often a given model has only finitely degenerate, homogeneous absorbing state that does not carry any of such information. However, there exists a number of models that have infinitely many absorbing states. When the control parameter(s) set the model in the absorbing phase and far from the critical point, an absorbing state is reached quite fast and we might expect that, in such a case, the absorbing state will be almost randomly selected among all the absorbing states of the model. In the following, an absorbing state reached as a consequence of the dynamics of the model and starting from a random initial configuration will be called a natural absorbing state. However, when the system is close to the critical point, the evolution towards the absorbing state is quite long and can be complicated. It is very likely that, in this case, natural absorbing states constitute only a certain fraction of all absorbing states.

Are there any quantities that would specify which absorbing states are selected by the model's dynamics? In the context of phase transitions it would be particularly interesting if such quantities would also contain information about the critical point of the model (e.g., its location and maybe some of its exponents). We are not aware that, such questions, which would certainly provide a better understanding of models with absorbing states, have been previously considered.

The aim of the present paper is to investigate the above questions by studying a recently introduced one-dimensional model with infinitely many absorbing states [3]. Critical behavior of this model seems to be closely related with that of some multispecies branching-annihilating random-walk models (BARW). Using Monte Carlo method we show that the natural absorbing states do contain important information about the critical behavior of the model. In particular, we show that certain characteristics of natural absorbing states exhibit power singularities upon approaching the critical point.

The model and some of its basic properties are described in Sec. II. Monte Carlo results that show that the structure of natural absorbing states contains information about criticality of the model are discussed in Sec. III. In Sec. IV, numerical findings are corroborated by random-walk arguments. In Sec. V we also look at the dynamics of our model as a certain minimum-finding procedure. Such an approach, when suitably generalized might serve as a general-purpose technique to solve some optimization problems. Finally, concluding remarks are made in Sec. VI.

II. MODEL AND ITS BASIC PROPERTIES

Our model is defined on a one-dimensional ladderlike lattice. For each bond between the nearest-neighboring sites we introduce a bond variable $w \in (-1,1)$ [4]. Introducing parameters r and s, we call a given site active when $w_1 w_2 |w_3|^s < r$, where w_1 and w_2 are intrachain bond variables connected with this site and w_3 is the interchain variable. Otherwise, this site is called nonactive. The model is

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driven by random sequential dynamics and when the active site is selected, we assign anew, with uniform probability, three neighboring bond variables. Nonactive sites are not updated. For a more detailed description of this model see Ref. [3].

Models of this kind might be used to describe coarse-grained biological evolution, where species (sites) mutate only when the interaction with other species puts to much pressure on it (i.e., when the site is active) [5]. In addition, the above model provides an example of a coupled system whose criticality is changed due to this coupling. Recently, related systems are drawing certain attention [6].

Monte Carlo simulations supported by some analytic considerations clarified the basic properties of this model, which can be briefly sumarized as follows [3]. For $r>0$ the model is in the active phase. For $r<0$ the model generates, at a finite rate, sites that are nonactive and that will remain in this state forever. Thus, the model, very quickly, enters an absorbing state and for $r<0$ the model is in the absorbing phase. The transition point $r=0$ is critical and some quantities exhibit a power-law behavior. For example, the order parameter ρ decays in time with an exponent δ

$$\rho(t)\sim t^{-\delta} \text{ with } \delta=0.5. \tag{1}$$

The average time τ needed to reach an absorbing state increases with the system size N as

$$\tau\sim N^{z} \text{ with } z=2. \tag{2}$$

Moreover, for $r\to 0^{+}$, the order parameter scales as $\rho\sim r^{\beta}$, while the transverse correlation length diverges as $\xi_{\perp}\sim r^{-\nu_{\perp}}$ with

$$\beta=\nu_{\perp}=1/s \text{ for } s\geqslant 1. \tag{3}$$

At $r=0$ the model's dynamics is particularly simple since in this case the coupling between chains (w_3) is irrelevant and only the sign of w_1w_2 determines (non)activity of a site. The dynamics of the model is thus similar to a certain branching-annihilating random-walk model, for which exponents (1) and (2) can be derived analytically [7].

At first sight, from Eq. (3) it seems that our model has continuously changing (with s) exponents β and $\nu_{\perp}$. However, it was shown [3] that this nonuniversality can be removed when the control parameter r is replaced by the reactivation probability [8]. Note that the critical behavior of this model is related with that of certain BARW models. Indeed, critical exponents (3) for $s=1$ are the same as in the multispecies BARW model without exclusion [9]. Moreover, exponents for $s=2$ correspond to the BARW model with hard-core exclusion [10,11]. Since nonuniversality of our model can be removed by an appropriate redefinition of the control parameter, it suggests that a similar situation might take place in BARW models. However, up to now, it seems that there is no simple relation between BARW models with and without exclusion. In addition, it would be interesting to generalize BARW models so that exponents β and $\nu_{\perp}$ would change continuously with a certain parameter that would interpolate between the case with and without exclusion.

III. STRUCTURE OF NATURAL ABSORBING STATES

In this section we examine natural absorbing states of the model for $r\leqslant 0$. First, let us define the following pseudoenergy e as

$$e=-\frac{1}{2N}\sum_{j} w_{1j}w_{2j}|w_{3j}|^{s}, \tag{4}$$

where summation is over all sites j of the model, w_{1j}, w_{2j}, and w_{3j} are three bond variables connected to the site j, and N is the linear size of the system. Active sites are those whose contributions to Eq. (4) exceeds $-r$. Another interesting quantity is the average value $\overline{e}(r)$ of e calculated for natural absorbing states.

Note that at criticality, $\overline{e}(0)$ can be calculated exactly. Indeed, in this case any of the absorbing states has the following structure: each chain has all intrachain variables w of the same sign and interchain variables can take arbitrary values. Moreover, for $r=0$ the dynamics of the model is such that it is only the sign of bond variables that matters while their values are irrelevant. It means that the only requirement for a site belonging to a certain absorbing state is that two intrachain variables connected to this site are of the same sign. Hence,

$$\begin{aligned}\overline{e}(0)&=-\frac{1}{2N}(2N)\int_0^1 w_1dw_1\int_0^1 w_2dw_2\int_{-1}^1\frac{1}{2}|w_3|^{s}dw_3\\&=-\frac{1}{4(s+1)},\end{aligned} \tag{5}$$

where the triple integral gives the single site contribution to $\overline{e}(0)$.

In the absorbing phase, the situation is not so simple and an analytic evaluation of $\overline{e}(r)$ is not possible. However, this quantity can be estimated by Monte Carlo simulations. For a given value of r, we start from a random bonds configuration and evolve the system until an absorbing state is reached. The average $\overline{e}(r)$ is then obtained by calculating e for several independent absorbing states.

Simulations were done for systems of size $N=1000$. For $r=0$, the analytical prediction was recovered with a great precision by averaging over approximately 10^3 independent absorbing states. As a check, the result of Eq. (5) for $s=1$, 2, and 4 was reproduced with the accuracy of $10^{-3}\sim 10^{-4}$.

Our main concern is, however, the off-critical behavior of $\overline{e}(r)$. For $r<0$ the requirement that all intrachain bonds have to be of the same sign is no longer necessary. As a result, absorbing states might have a more complicated structure, hence we were not able to calculate $\overline{e}(r)$ exactly. However, since for $r<0$ the dynamics leads to an absorbing state much faster than at criticality, we expect that $\overline{e}(r<0)>\overline{e}(0)$ and that $\overline{e}(r)$ is a decreasing function of r. Indeed, the dynamics searches for states with small pseudoenergy. This behavior is confirmed by the numerical simulations. In addition, as

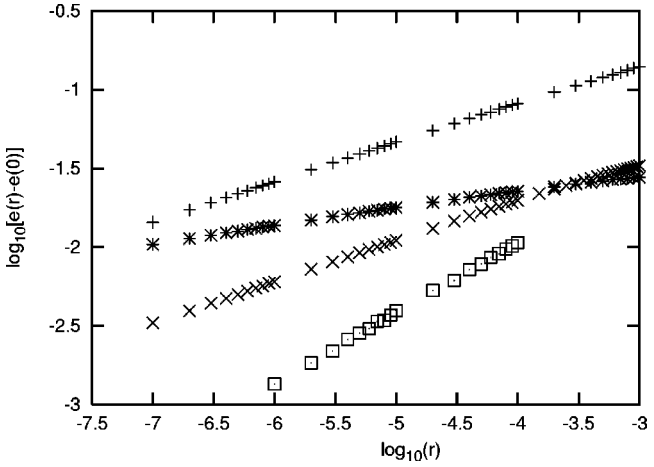

FIG. 1. The log-log plot of $\overline{e}(r)-\overline{e}(0)$ as a function of r for $s=1$ (□), 2 (×), and 4 (⋆). For $s=2$ we also plot the density of permanently nonactive sites (+).

shown in Fig. 1, our results indicate that $\overline{e}(r)$ has a power-law singularity at $r=0$ and for vanishing r behaves as

$$\overline{e}(r)-\overline{e}(0)\sim r^{p}, \tag{6}$$

where p is an s-dependent exponent.

The least square fit to these data (using the ten closest datapoints to $r=0$) leads to the following exponents: $p=0.46, 0.245$, and 0.128 for $s=1, 2$, and 4, respectively. These values of p suggest that the true exponents (for infinite systems) are $p=1/2s$. In the following section we present a random-walk argument that supports this claim.

IV. RANDOM-WALK ARGUMENT

First let us recall certain properties of this model that hold for $r<0$. As it was already shown [3] in this case the model generates sites that remain permanently nonactive. Indeed, if after updating certain active site the interchain bond w_3 satisfies the condition

$$|w_3|^{s}<-r, \tag{7}$$

then both sites connected to w_3 remain permanently nonactive. Since there is a finite probability to satisfy Eq. (6), for $r<0$ activity in the system quickly dies out and the model is in the absorbing phase.

Now, to explain the observed scaling of $\overline{e}(r)-\overline{e}(0)$, let us assume that r is negative and very close to zero. As it was already discussed, at $r=0$ an absorbing state is composed of ''ferromagnetic'' chains. We think that the following scenario describes the dynamics for $r<0$. For early times, the evolution of the model resembles the $r=0$ case and the system develops larger and larger ''ferromagnetic'' domains (coarsening). At the late stage the evolution of the system is mainly governed by the dynamics of the domain walls. For $r=0$ the system would coarsen until the fully ''ferromagentic'' state was reached. However, for $r<0$, the activity between domains might die out quicker, mainly due to the possibility of creation of permanently absorbing sites (in the following we will present some Monte Carlo data supporting this assumption).

Using this assumption we estimate the number of permanently nonactive sites N_p as follows. First, let us notice that for $r=0$, the coarsening proceeds with the exponent $\delta=0.5$ [3]. It means that the number of active sites $N_a(t)$ in the system scales for large time as $N_a(t)\sim Nt^{-0.5}$. It is well known that the time needed to form a domain of size l by coarsening (and for a dynamics with nonconservation law) scales as l^2 (random-walk argument [12]). Next, we estimate the total number of updates needed to create domains of the typical size l as

$$\int_0^{l^2} N_a(t)\,dt\sim Nl. \tag{8}$$

From Eq. (8) the probability of creating of a permanently nonactive site scales as $(-r)^{1/s}$. It means that the typical size l that scales as the inverse of the density of permanently nonactive sites behaves as

$$l\sim\frac{N}{Nl|r|^{1/s}}. \tag{9}$$

In the above relation we assume that the number of permanently nonactive sites scales as the product of the total number of updates (8) with the probability of their creation $(\sim|r|^{1/s})$. Using Eq. (9) we obtain that the density of permanently nonactive sites scales as

$$1/l\sim|r|^{1/(2s)}. \tag{10}$$

The last relation explains the scaling relation $\overline{e}(r)-\overline{e}(0)\sim|r|^{1/(2s)}$ observed in Monte Carlo simulations of the previous section. Indeed, for r negative and close to zero the only ''excitations'' above the $\overline{e}(0)$ are due to the permanently nonactive sites and this contribution should be proportional to their density.

To confirm our scaling arguments we measured the density of interchain bonds satisfying $|w_3|<|-r|^{1/s}$. For $s>1$ majority of permanently nonactive sites is created on such bonds. Our results, for $s=2$ presented in Fig. 1, confirm that the density of such sites scales as $|r|^{1/(2s)}$. At the same time this confirms the consistency of the arguments presented in this section.

V. DYNAMICS AS A MINIMUM-FINDING PROCEDURE

In this section we consider the dynamics of our model as a procedure to minimalize e: active sites are only those whose contributions to e exceed $-r$. Of course, the pseudoenergy e is a meaningful quantity also in the active phase. Thus, one can ask the following question: In which phase the dynamics finds more optimal solution? In the active phase the dynamics cannot inactivate all sites beacause the requirements for that are too tight. Thus, even when a ''good'' set of sites is found it gets destroyed by neighboring active sites. As a result a finite fraction of sites remains active with a

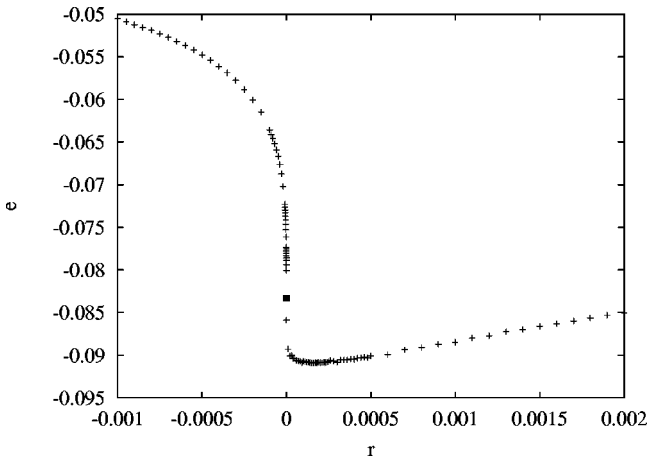

FIG. 2. The pseudoenergy e as a function of r for $s=2$. In the absorbing phase ($r<0$) the results are averaged over 10^3 absorbing states ($L=10^3$). In the active phase ($r>0$) we made an ordinary steady-state averaging ensuring that the presented results are size independent (for r close to zero simulations were made for $L=10^5$). The black square denotes the exact value (5).

relatively large contributions to e. In the absorbing phase a ''good'' set has a much larger probability to survive but the condition for being inactive are milder now and, as a result, worse solutions are also being found. Competition of all these effects implies that the r dependence of the pseudoenergy is rather difficult to predict.

To approach this problem we measured e also in the active phase and our results are shown in Fig. 2. In the absorbing phase we can see, the already analyzed, singular behavior with minimum for $r=0$. More interesting is the behavior in the active phase. First, we can see that there is a (narrow) range of r where in the steady state e is lower than in the absorbing phase. Most likely, however, for $r\to 0^+$ the pseudoenergy e continuously approaches the $r=0$ value (5). The lowest-e solution is found for $r\sim 0.000\,18$ that is very close to the critical point but not at the critical point.

In the field of combinatorial optimization, one frequently encounters problems similar to minimization of functions like Eq. (4). For example, the traveling salesman problem [13] or number partitioning [14] are equivalent to minimization of certain spin-glass-like Hamiltonians. Main techniques to deal with such problems are usually simulated annealing [15], genetic algorithms [16], or their hybrids. In principle it should be possible to extend our approach to solve some other optimization problems too. Having an energy function we should define active sites as those whose contributions to this energy exceed certain value and then evolve the system similarly to the present model. Applicability and efficiency of this approach to deal with more typical optimization problems is, however, left as a future problem. In addition, we hope that such a method might reveal new connections between statistical mechanics of models with absorbing states and computational techniques.

VI. CONCLUSIONS

In this work, we have shown that natural absorbing states in a certain model contain important information about the critical point of the model. This information is encoded in the static properties of these states. We should mention that natural absorbing states are well known to contain information about dynamic properties of models with infinitely many absorbing states. In particular, characteristics of the so-called spreading are known to have power-law singularities at the critical point and the corresponding exponents exhibit certain universality.

An extension of this work is to check whether our results are applicable to other models with infinitely many absorbing states. Such models appears in various contexts ranging from catalysis [17] to self-organized criticality [18] and biological evolution [5]. In some cases, the absorbing states are expected to be quite complex [19] and some indications of criticality might be hidden in their static structure. Unfortunately, we do not know which quantities would exhibit these critical singularities. In particular, it is not obvious to us that in other models there exist permanently nonactive sites that most likely are responsible for the singularities observed in our model. Another question is what are the exponents characterizing these critical singularities (provided they exist). In our case, the corresponding exponent $[1/(2s)]$ equals to half of the exponent $\beta(=1/s)$ and it would be interesting to check whether similar relation holds for other models.

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